Math 245C Lecture 24 Notes

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1 Distributions and Smooth Urysohn's Lemma

1.1 Distributions

Throughout this section, $U \subseteq \mathbb{R}^n$ is an open set.

Definition 1.1. If $E \subseteq \mathbb{R}^n$, $C_c^\infty(E)$ is the set of $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}(\phi) \subseteq E$.

We endow $C_c^\infty(U)$ with the following topology: $(\phi_j)_{j \in \mathbb{N}} \subseteq C_c^\infty(U)$ converges to $\phi \in C_c^\infty(U)$ if there exists a compact $K \subseteq U$ such that

- $\operatorname{supp}(\phi_j) \subseteq K$ for all j,
- $\partial^\alpha \phi_j \to \partial^\alpha \phi$ uniformly on K for all $\alpha \in \mathbb{N}^n$.

Definition 1.2. Let X be a locally convex topological vector space. A linear operator $T : C_c^\infty(U) \to X$ is **continuous** if for each compact $K \subseteq U$, $T|_{C_c^\infty(K)}$ is continuous.

Definition 1.3. Let U' be an open subset of $\mathbb{R}^n$. A linear operator $T : C_c^\infty(U) \to C_c^\infty(U')$ is **continuous** if for each compact $K \subseteq U$, there exists a compact $K' \subseteq U'$ such that $T(C_c^\infty(K)) \subseteq C_c^\infty(K')$, and $T : C_c^\infty(K) \to C_c^\infty(K')$ is continuous.

Definition 1.4. If $T : C_c^\infty(U) \to \mathbb{R}$ is linear and continuous, we say that T is a **distribution** on U and write $T \in \mathcal{D}'(U)$.[1]

Definition 1.5. If $V \subseteq U$ and $T, S \in \mathcal{D}'(U)$, we say that $T = S$ on V if $T(\phi) = S(\phi)$ for all $\phi \in C_c^\infty(V)$.

Definition 1.6. A sequence $(T_j)_{j \in \mathbb{N}} \subseteq \mathcal{D}'(U)$ **converges** to $T \in \mathcal{D}'$ if $\lim_{j \to \infty} T_j(\phi) = T(\phi)$ for all $\phi \in C_c^\infty(U)$.

That is, $\mathcal{D}'(U)$ is endowed with the weak* topology.

[1] This notation is because some people call $\mathcal{D} := C_c^\infty(U)$ and denote the dual by $'$.

Example 1.1. Let $f \in L^1_{\text{loc}}(U)$. Define

$$T(\phi) = \int_U f\phi\, dx, \qquad \phi \in C_c^\infty(U).$$

This is a distribution.

Example 1.2. Let μ be a Radon measure on U. Define

$$T(\phi) = \int_U \phi(x)\, d\mu(x).$$

For example, let $x_0 \in U$, and $\mu = a\delta_{x_0}$. Set

$$T(\phi) = a\phi(x_0) = \int_U \phi(x)\, d\mu(x).$$

This is a distribution.

Notation: If $\phi : \mathbb{R}^n \to \mathbb{R}$, set $\tilde{\phi}(x) = \phi(-x)$.

Proposition 1.1. *Let $f \in L^1(\mathbb{R}^n)$. For each $t > 0$, set $f_t(x) = t^{-n}\phi(x/t)$ for $x \in \mathbb{R}^n$. Assume that $\int_{\mathbb{R}^n} f(x)\, dx = 1$. Define*

$$T_t(\phi) = \int_{\mathbb{R}^n} f_t(x)\phi(x)\, dx.$$

Then $T_t \to \delta_0$ in $\mathcal{D}'(\mathbb{R}^n)$; that is, $T_t \to T_0$, where $T_0 = \delta_0$.

Remark 1.1. Often, people will view f_t as its distribution T_t and call the distribution f_t.

Proof. Let $\phi \in C_c^\infty(\mathbb{R}^n)$. Observe that

$$T_t(\phi) = \int_{\mathbb{R}^n} f_t(x)\tilde{\phi}(0 - x)\, dx = f_t * \phi(0).$$

So we have

$$\lim_{t\to 0} T_t(\phi) = \lim_{t\to 0} f_t * \tilde{\phi}(0) = \tilde{\phi}(0) = \phi(0).$$ □

1.2 Smooth Uryson's Lemma

Proposition 1.2 (extension of Urysohn's lemma)**.** *Let $K \subseteq \mathbb{R}^n$ be compact, and let $U \subseteq \mathbb{R}^n$ be an open set containing K. Then there exists $\phi \in C_c^\infty(\mathbb{R}^n, [0, 1])$ such that $\phi|_K = 1$ and* $\text{supp}(\phi) \subseteq U$.

Remark 1.2. Urysohn's lemma is the case where we do not assume that ϕ is smooth.

Proof. Let $\rho \in C_c^\infty(\mathbb{R}^n)$ be such that $\rho \geq 0$, $\operatorname{supp}(\rho) \subseteq \overline{B_1(0)}$ and $\int_{\mathbb{R}^n} \rho(x)\, dx = 1$. Set $\rho_t(x) = t^{-n}\rho(x/t)$ for $t > 0$ and $x \in \mathbb{R}^n$. By Urysohn's lemma, there is a $g \in C_c(\mathbb{R}^n, [0,1])$ such that $g|_{K_\varepsilon} = 1$, $\operatorname{supp}(g) \subseteq U_\varepsilon$, where $K_\varepsilon = \{x \in \mathbb{R}^n : \operatorname{dist}(x, K) \leq \varepsilon\}$ and $U_\varepsilon = \{x \in U : \operatorname{dist}(x, U^c) > \varepsilon\}$. As K is compact, let $\delta = \operatorname{dist}(K, U^c) > 0$. If $0 < \varepsilon < \delta$¡ then $K \subseteq U_\varepsilon$, K_ε is compact, and U_ε is open. Let $\phi = \rho_{\delta/4} * g$, and let $\varepsilon = \delta/4$. Since $\rho_{\delta/4} \in C_\infty(\mathbb{R}^n)$, we have $\phi \in C^\infty(U)$. Note that

$$\phi(x) = \int_{\mathbb{R}^n} \rho(y/\varepsilon)\frac{1}{\varepsilon^n} g(x-y)\, dy = \int_{B_\varepsilon(0)} \rho_\varepsilon(x) g(x-y)\, dy.$$

If $x \in K$ and $|y| < \varepsilon$¡ the $x - y \in K_\varepsilon$, and so $g(x-y) = 1$. Hence,

$$\phi(x) = \int_{B_\varepsilon(0)} \rho_\varepsilon(x)\, dx = 1.$$

If $x \notin U^\varepsilon$, then $g(x-y) = 0$ if $|y| < \varepsilon$. Hence, $\phi(x) = 0$. □